Finite Strain Consolidation

Numerical Methods in Geotechnical Engineering

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1. Introduction

Terzaghi's theory, although widely used, is based on assumptions that are rarely met in practice. The most significant of these assumptions is that the strains are small in the porous media in which consolidation takes place. Often there will be significant strains in the media and therefore new theory must be developed. It has been shown that the compressibility for a saturated soil is a non-linear function of the effective stress state of the soil. This must be accounted for in the theory if the solution is to be correct. Terzaghi's theory also assumes that the permeability of the soil remains constant during the consolidation process. When large strains take place, the soil pores are squeezed and there is a resulting decrease in the void ratio of the soil. A smaller void ratio means that water has less room to flow and there is a resulting decrease in the permeability of the soil.

There have been attempts made to extend Terzaghi's theory of consolidation to take account of large strains (Richart, 1957; Lo, 1960; Davis & Raymond, 1965; Janbu, 1965; Barden and Berry, 1965). These theories are still based on essentially small strain theory and therefore have limitations.

It was the desire of the author to develop the theory of large strain consolidation and solve the theory using a finite difference technique. Once the finite difference model was working sufficiently, results would be compared to existing consolidation computer models to verify accuracy.

A paper by F.C. Townsend on "Large Strain Consolidation Predictions" provided a comparison of the most common computer programs used in the prediction of consolidation rates. These currently available consolidation programs have been used extensively by the Florida phosphate mining industries for compliance with regulatory activities. The paper by F.C. Townsend was written to provide a forum by which the different prediction methods might be compared.

In comparison of the different models, it was found by F.C. Townsend that predictions varied from program to program. This difference was attributed to differences in the coordinate system used (Eularian vs. Lagrangian), the dependent variable selected (void ratio vs. pore-water pressure), the finite difference solution technique employed, (implicit vs. explicit), or the solution methodology used (finite difference vs. finite element). It was the purpose of F.C. Townsend to provide a standard of comparison for these programs and gain understanding of their advantages and limitations. This also provided a basis for comparison to the finite difference consolidation model developed by the author.

In addition to comparison to other finite strain models, comparison will also be made to Terzaghi's standard consolidation equation. The progress at which consolidation settlement occurs as well as the rate pore-water pressures dissipate will be examined.

2. Theory

Differences in solutions between computer programs can be traced back to the theory that the solutions are based on. Theoretical differences in solutions can be categorized into three areas: differences in coordinate system used, in the dependent variable solved for, and in the solution methodology.

2.1 Coordinate Systems

The most commonly used coordinate system in geotechnical engineering is the Eularian system where material deformations are related to planes fixed in space. This fixed plane is commonly referred to as a datum. Distances are then measured relative to this datum. Properties of the Representative Elementary Volume (REV) are referenced to a specific distance from the datum. Terzaghi's consolidation theory which is based on this type of system then assumes that both the size and position of the element remain the same over time. Any deformations that do take place in the soil element are assumed to be small in comparison to the size of the element. This can be visualized by thinking of a piezometer installed a fixed distance from a datum. Using infinitesimal strain, it is assumed that the distance from the datum to the piezometer will always remain the same.

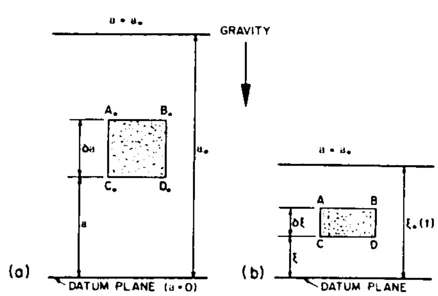


FIG. 1. Lagrangian and convective coordinates: (*a*) initial configuration at $t = 0$; (*b*) configuration at time t.

Figure 1 - Lagrangean and convective coordinates (Gibson)

With finite strain consolidation, the deformations are large compared to the thickness of the compressible layer. This means that properties referenced to a certain y-coordinate may suddenly be outside the element they refer to if deformations are large enough. A system must be found that deforms with the material particles. This would mean in the above example that the piezometer is always surrounded by the same material. Such a coordinate system can be either a convective system or a lagrangean coordinate system as shown in Figure 1. When the soil element deforms, the location and size of the soil element changes and this is reflected by the changing coordinates. Changes with time can be related to the either the convective system (ξ,t) or to the lagrangian system (a,t).

2.2 The Dependent Variable

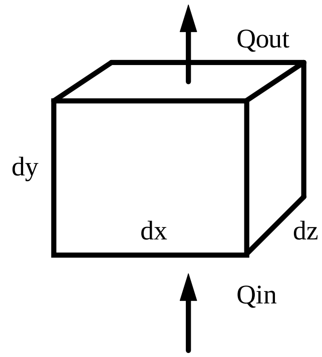
Solution methods can also vary by the dependent variable in the governing equation. Typical dependent variables include pore-water total head, pore-water pressure, effective stress, and void ratio. Terzaghi's consolidation equation used pore-water pressure as the dependent variable. With pore-water pressure as the dependent variable, boundary conditions were easily specified. No flow and constant pressure boundary conditions can both be specified with ease. The primary drawback of using pressure or head as a dominant variable seems to be that in finite strain formulations the resulting equation is highly non-linear. This results in difficulties in finding a solution method to handle the non-linear nature of the equation. Davis & Raymond were the first two people to develop a consolidation equation taking into account variations of permeability and compressibility of the soil. This equation was based on effective stress and it was found that pore-water pressure dissipation occurred at a slower rate than Terzaghi's equation predicted when deformation of the soil was taken into account. Work was later done by Robert E. Gibson of King's College in UK on consolidation and an equation was developed that used void ratio as the dependent variable. Using void ratio seemed to result in an equation that was easier to solve and thus resulted in the popularity of Gibson's equation.

Due to reasonable difference in the different equations describing finite strain consolidation, the following sections will contain derivations of the equations to illustrate in detail the differences between the classical derivations and the finite strain consolidation derivation presented in this paper.

2.2.1 Classical Consolidation Equation

The most common consolidation equation that takes into account the non-linear soil properties of permeability and compressibility is presented below. Terzaghi's classic consolidation equation can be simplified out of this derivation.

REV



Assuming adherence to Darcy's Law

$$q = -k\frac{dh}{dy}$$

Continuity dictates that

$$\frac{\partial M}{\partial t} = Q_{in} - Q_{out}$$

where: M = mass (kg)
t = time (s)
Qin = mass flow in (kg/s)
Qout = mass flow out (kg/s)

but $M = \rho_w \,.\, V_w$ where: ρ_w = density of water (kg/m^3)
V_w = volume of water (m^3)

and $Q_{in} = \rho_w\, q_y\, dx\, dz$ where: q_x = flow of water per unit area (m^3/s/m^2)

also
$$Q_{out} = Q_{in} + \Delta Q$$
$$= \rho_w . q_y . dx. dz + \frac{\partial}{\partial y}(\rho_w . q_y . dx. dz). dy$$

now substitute into the continuity equation

$$\frac{\partial(\rho_w V_w)}{\partial t} = \rho_w . q_y . dx. dz - \left[\rho_w . q_y . dx. dz + \frac{\partial}{\partial y}(\rho_w . q_y . dx. dz) dy\right]$$

this reduces to

$$\frac{\partial(\rho_w V_w)}{\partial t} = -\frac{\partial}{\partial y}(\rho_w . q_y . dx. dz) dy$$

but $dy \,.\, dz \,.\, dx = V_t$ where: V_t = total volume (m^3)

so with densities canceling we have

$$\frac{\partial\left(V_w / V_t\right)}{\partial t} = -\frac{\partial q_y}{\partial y} \qquad (2.1)$$

Now according to Darcy's Law

$$q_y = -k_y \frac{dh}{dy}$$

where: k_y = permeability (m/s)

h = head = $\dfrac{u}{\rho_w g} + z$ (m)

u = pore-water pressure (kPa)

g = acceleration of gravity (m/s^2)

z = elevation (m)

Substituting into (2.1)

$$\frac{\partial\left(V_w / V_t\right)}{\partial t} = \frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial y}\right)$$

but since k varies with depth k = fn(x) or k(x)
so by the chain rule

$$\frac{\partial\left(V_w / V_t\right)}{\partial t} = \frac{\partial k(y)}{\partial y}\frac{\partial h}{\partial y} + k(y)\frac{\partial^2 h}{\partial y^2} \qquad (2.2)$$

Now a change in volume means a change in the storage of the REV and we must find a way to represent this

$$\varepsilon_v = \frac{\Delta V}{V_t}$$

where: ε_v = volumetric strain (m^3/m^3)

$$-m_v = \frac{\Delta \varepsilon_v}{\Delta \sigma'}$$

where: m_v = coefficient of volume change (m^3/kN)

so $$-m_v = \frac{\left(\Delta V / V_t\right)}{\Delta \sigma'}$$

assume all change in volume is due to water loss (this is the case for saturated soils)

but $\Delta\sigma' = \Delta\sigma_t - \Delta u_w$

and we assume that total stress does not change

so $\Delta\sigma' = -\Delta u_w$

so $$m_v = \frac{\partial\left(V_w / V_t\right)}{\partial u_w}$$

$$\partial\left(V_w / V_t\right) = m_v \partial u_w \qquad \text{and} \qquad u = (h - y)\rho_w g$$

$$u = h\rho_w g - y\rho_w g$$

Now differentiate with respect to time

$$\frac{\partial\left(V_w / V_t\right)}{\partial t} = m_v \frac{\partial u_w}{\partial t}$$

Since the rest of the equation is in total head, the pore-water pressure term will be converted

$$\frac{\partial h}{\partial t} = \frac{\partial u}{\partial t}\frac{1}{\gamma_w} + \frac{\partial y}{\partial t}$$

And if infinitesimal strain is assumed, then the term $\partial y/\partial t = 0$ so

$$\frac{\partial\left(V_w / V_t\right)}{\partial t} = \gamma_w m_v (y).\frac{\partial h}{\partial t}$$

Substituting into (2.2) we get

$$\gamma_w m_v (y).\frac{\partial h}{\partial t} = \frac{\partial k(y)}{\partial y}\frac{\partial h}{\partial y} + k(y)\frac{\partial^2 h}{\partial y^2}$$

$$\frac{\partial h}{\partial t} = \frac{1}{\gamma_w m_v (y)}\left[\frac{\partial k(y)}{\partial y}\frac{\partial h}{\partial y} + k(y)\frac{\partial^2 h}{\partial y^2}\right] \qquad (2.3)$$

This can also be illustrated in terms of pore-water pressure by differentiating the following expression:

$$h = \frac{u}{\gamma_w} + y$$

$$\frac{\partial h}{\partial y} = \frac{\partial u}{\partial y}\frac{1}{\gamma_w} + 1 \qquad \text{also} \qquad \frac{\partial h}{\partial t} = \frac{\partial u}{\partial t}\frac{1}{\gamma_w} + \frac{\partial y}{\partial t}$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{\partial^2 u}{\partial y^2}\frac{1}{\gamma_w}$$

If we assume that $\partial y/\partial t=0$ then we have

$$\frac{\partial h}{\partial t} = \frac{\partial u}{\partial t}\frac{1}{\gamma_w}$$

And substituting these equations into equation (2.3) we have

$$\frac{\partial u_w}{\partial t} = \frac{1}{\gamma_w m_v(y)}\left[\frac{\partial k(y)}{\partial y}\frac{\partial u_w}{\partial y} + \gamma_w \frac{\partial k}{\partial y} + k(y)\frac{\partial^2 u_w}{\partial y^2}\right]$$

If permeability does not change with depth then $\partial k/\partial y = 0$

so $$\frac{\partial u_w}{\partial t} = \frac{k(y)}{\gamma_w m_v(y)}\frac{\partial^2 u_w}{\partial y^2}$$

Then if k and m_v do not vary with depth and we set $c_v = \dfrac{k}{m_v \gamma_w}$ then we have Terzaghi’s standard consolidation equation

$$\frac{\partial u_w}{\partial t} = c_v \frac{\partial^2 u_w}{\partial y^2} \qquad (2.4)$$

Equation (2.4) has been used extensively to predict the dissipation of pore-water pressures in soil beneath an applied load. As can be seen from the derivation, a number of assumptions have been made in this derivation. To randomly apply equation (2.4) to all pore-water pressure problems would be a gross error. In fact, most of the assumptions made in the derivation of this final equation contradict the field conditions present. Permeability typically varies with void ratio. The coefficient of compressibility m_v varies with regards to effective stress which changes very dynamically with current pore-water pressures. If mass flows are important, then the equation used must include total head or else the solution will be wrong. Therefore equation (2.4) is often the better solution to consolidation problems encountered in the field.

2.2.2 Derivation of Davis & Raymond Consolidation Equation

In the Davis & Raymond equation, effective stress is used as the dominant variable. It was found that the dissipation of pore-water pressures was slower using the Davis & Raymond equation than Terzaghi’s formulation. It was also shown that the degree of settlement is identical with the Terzaghi one-dimensional theory but the amount of maximum pore-water pressure dissipation depends on the loading increment ratio. This is shown in Figure 2.

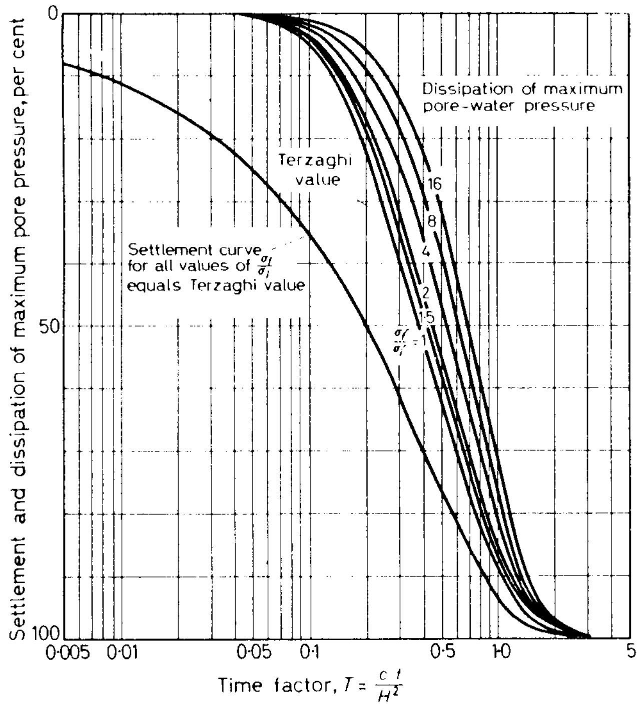
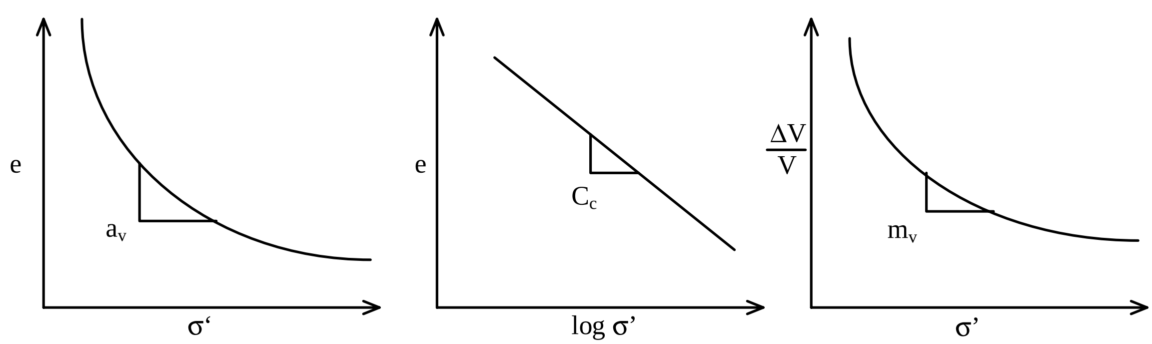


Figure 3.43. Solution for percentage settlement and dissipation, respectively, of maximum pore pressure (After Davis and Raymond,

Figure 2 - Comparison of Terzaghi and Davis & Raymond consolidation solution (Lee, p. 150)

The following derivation shows the mathematical techniques and assumptions used in arriving at their solution.

The three most common way of describing how a soil deforms are as follows:



This leads to the following equations describing saturated volume change

$$-a_v = \frac{\Delta e}{\Delta \sigma'} \qquad C_c = \frac{-\Delta e}{\log \dfrac{\sigma'_2}{\sigma'_1}} \qquad -m_v = \frac{\Delta V / V}{\Delta \sigma'}$$

By combining these equations we get

$$m_v = \frac{Cc}{1+e_o}(0.434)\frac{1}{\sigma'}$$

set $A = \frac{Cc}{1+e_o}(0.434)$ so $m_v = \frac{A}{\sigma'}$

Now derive according to continuity and net velocity

$$\text{net velocity} = \frac{\partial v}{\partial y}dy \qquad v = -k\frac{\partial h}{\partial y} = -\frac{k}{\gamma_w}\frac{\partial u}{\partial y}$$

but now also

$$Cv = \frac{k}{m_v \gamma_w} = \text{constant}(\text{assume})$$

so $k = C_v\, m_v\, \gamma_w$

$$\frac{\partial v}{\partial y}dy = \frac{\partial}{\partial y}\left(\frac{-k}{\gamma_w}\frac{\partial u}{\partial y}\right)dy$$

$$\frac{\partial v}{\partial y}dy = -\frac{\partial}{\partial y}\left(\frac{C_v m_v \gamma_w}{\gamma_w}\frac{\partial u}{\partial y}\right)dy$$

$$\frac{\partial v}{\partial y}dy = -C_v A\frac{\partial}{\partial y}\left(\frac{1}{\sigma'}\frac{\partial u}{\partial y}\right)dy$$

Now differentiate the expression using the product rule

$$\frac{\partial v}{\partial y}dy = -C_v A\left[\frac{1}{\sigma'}\frac{\partial^2 u}{\partial y^2} - \left(\frac{1}{\sigma'}\right)^2\frac{\partial u}{\partial y}\frac{\partial \sigma'}{\partial y}\right]dy$$

Now introduce strain in an element

$$\frac{\partial \varepsilon}{\partial t} = \frac{A}{\sigma'}\frac{\partial \sigma'}{\partial t}$$

And now this is related to the equation describing change in velocity

$$\frac{\partial v}{\partial y}dy = \frac{\partial \varepsilon}{\partial y}dy$$

$$-C_v A\left[\frac{1}{\sigma'}\frac{\partial^2 u}{\partial y^2} - \left(\frac{1}{\sigma'}\right)^2 \frac{\partial u}{\partial y}\frac{\partial \sigma'}{\partial y}\right] = \frac{A}{\sigma'}\frac{\partial \sigma'}{\partial t}$$

$$\frac{\partial \sigma'}{\partial t} = -C_v \sigma'\left[\frac{1}{\sigma'}\frac{\partial^2 u}{\partial y^2} - \left(\frac{1}{\sigma'}\right)^2 \frac{\partial u}{\partial y}\frac{\partial \sigma'}{\partial y}\right]$$

The above equation ends up being a slightly non-linear equation describing the consolidation process. The results predicted from this equation were reasonably close to the Terzaghi formulation for the reason that the coefficient of consolidation was found to remain reasonably constant with varying stress levels. As the effective stress increased, an increase in the confined modulus was balanced by a decrease in the coefficient of permeability resulting in a reasonably constant coefficient of consolidation. This is shown in Figure 3. While including the nonlinear properties of permeability and compressibility, this formulation still assumes infinitesimal strains.

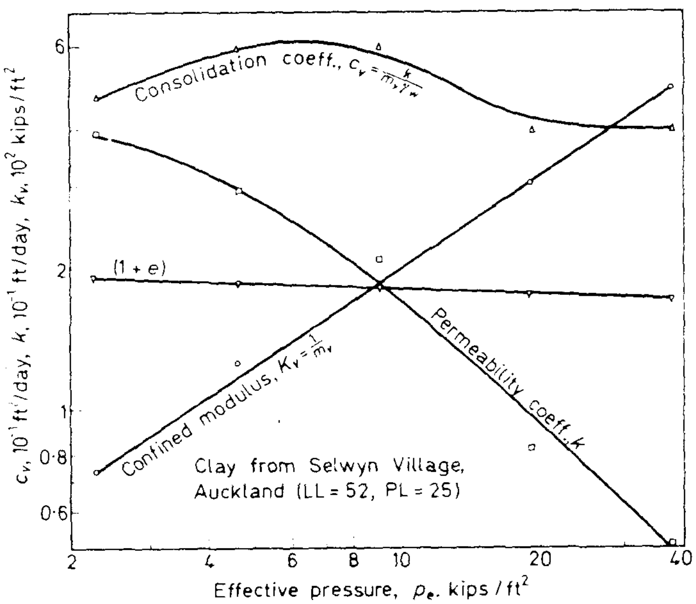


Figure 3.33. Typical oedometer results on normally consolidated soil (Davis, unpublished)

Figure 3 - Variation of c_v with changing k and m_v (Lee, p. 137)

2.2.3 Derivation of Gibson's Consolidation Equation

The following derivation is taken from "The Theory of One-Dimensional Consolidation of Saturated Clays" by Robert E. Gibson. Assuming the density of pore fluid and solids are both constant, vertical equilibrium requires that

$$\frac{\partial\sigma}{\partial z} \pm (e\rho_f + \rho_s) = 0$$

Note: the sign is positive if measured with gravity and negative if measured against gravity.

In addition, the equilibrium of the pore-fluid requires that

$$\frac{\partial p}{\partial z} - \frac{\partial u}{\partial z} \pm \rho_f \frac{\partial \xi}{\partial z} = 0$$

where p is the pore water pressure, and u is the excess pore water pressure. ξ is a Lagrangean coordinate describing the height of the soil element. If

$$\frac{\partial}{\partial z}\left[\frac{e(v_f - v_s)}{1+e}\right] + \frac{\partial e}{\partial t} = 0$$

then continuity of pore fluid flow is ensured, where v_f and v_s are the velocities of the fluid and solid phases relative to the datum plane.

Finally, Darcy's law requires that

$$\frac{e(v_f - v_s)}{k} \pm (1+e) + \frac{1}{\rho_f}\frac{\partial p}{\partial z} = 0$$

where p is the pore water pressure and k is the coefficient of permeability.

If the soil skeleton is homogeneous and possesses no creep effects and the consolidation is monotonic, then the permeability k may be expected to depend on the void ratio alone, so that

$$k = k(e)$$

while the vertical effective stress

$$\sigma' = \sigma - p$$

controls the void ratio, so that

$$\sigma' = \sigma'(e)$$

Equations are then combined to yield the following equation for the void ratio:

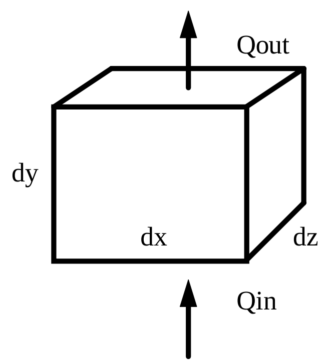
$$\pm\left(\frac{\rho_s}{\rho_f} - 1\right)\frac{\partial}{\partial e}\left[\frac{k(e)}{1+e}\right]\frac{\partial e}{\partial z} + \frac{\partial}{\partial z}\left[\frac{k(e)}{\rho_f(1+e)}\frac{\partial \sigma'}{\partial e}\frac{\partial e}{\partial z}\right] + \frac{\partial e}{\partial t} = 0$$

which appears to be highly nonlinear. This equation can be rendered linear, while retaining the non-linearity of the permeability and the compressibility, by examining the relationship between soil properties. It can also be shown that the above equation can be simplified back to the conventional, linear, infinitesimal strain equation (Terzaghi, 1924) and a non-linear version which also assumes infinitesimal strains (Davis & Raymond 1965; Raymond 1969)

2.2.4 Derivation of Finite Strain Consolidation Equation

The following derivation was obtained by attempting to correct certain deficiencies in other infinitesimal strain formulations. Although not as complete a formulation as Gibson's large strain equation, it was attempted to formulate and solve a modified equation and compare results to standard data. For comparison purposes, the complete derivation of the modified consolidation equation is presented below.

REV



Assuming adherence to Darcy's Law

$$q = -k\frac{dh}{dy}$$

Continuity dictates that

$$\frac{\partial M}{\partial t} = Q_{in} - Q_{out}$$

where: M = mass (kg)

t = time (s)
Qin = mass flow in (kg/s)
Qout = mass flow out (kg/s)

but $M = \rho_w \,.\, V_w$

where: ρ_w = density of water (kg/m^3)
V_w = volume of water (m^3)

and $Q_{in} = \rho_w \, q_y \, dx \, dz$

where: q_x = flow of water per unit area (m^3/s/m^2)

also

$$\begin{aligned} Q_{out} &= Q_{in} + \Delta Q \\ &= \rho_w.q_y.dx.dz + \frac{\partial}{\partial y}(\rho_w.q_y.dx.dz).dy \end{aligned}$$

now substitute into the continuity equation

$$\frac{\partial(\rho_w V_w)}{\partial t} = \rho_w.q_y.dx.dz - \left[\rho_w.q_y.dx.dz + \frac{\partial}{\partial y}(\rho_w.q_y.dx.dz)dy\right]$$

this reduces to

$$\frac{\partial(\rho_w V_w)}{\partial t} = -\frac{\partial}{\partial y}\left(\rho_w . q_y . dx . dz\right)dy$$

but dy . dz . dx = V_t where: V_t = total volume (m^3)

so with densities canceling we have

$$\frac{\partial\left(V_w / V_t\right)}{\partial t} = -\frac{\partial q_y}{\partial y} \qquad (2.5)$$

Now according to Darcy's Law

$$q_y = -k_y \frac{dh}{dy}$$

where: k_y = permeability (m/s)

h = head = $\dfrac{u}{\rho_w g} + z$ (m)

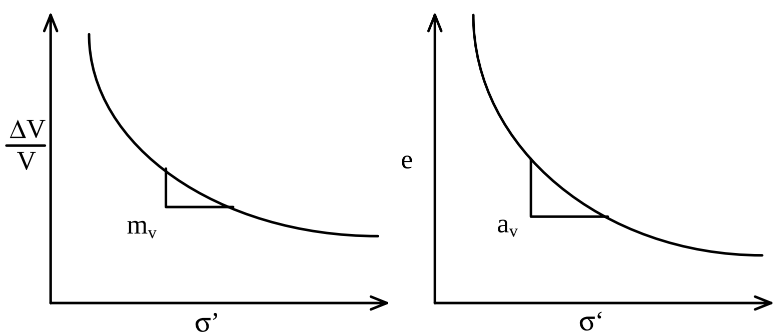
u = pore-water pressure (kPa)

g = acceleration of gravity (m/s^2)

z = elevation (m)

Substituting into (2.5)

$$\frac{\partial\left(V_w / V_t\right)}{\partial t} = \frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial y}\right)$$



This leads to the following equations describing saturated volume change

$$-a_v = \frac{\Delta e}{\Delta \sigma'} \qquad -m_v = \frac{\Delta V / V}{\Delta \sigma'}$$

Now differentiate with respect to time so

$$\frac{\partial\left(V_w / V_t\right)}{\partial t} = -m_v \frac{\partial \sigma'}{\partial t}$$

When the two equations relating volume change are combined we have

$$-m_v \frac{\partial \sigma'}{\partial t} = \frac{\partial}{\partial y}\left(k \frac{\partial h}{\partial y}\right)$$

The following substitutions are then made in place of effective stress

$$\frac{\partial \sigma'}{\partial t} = \frac{\partial \sigma}{\partial t} - \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial t} = \gamma_w \frac{\partial h}{\partial t} - \gamma_w \frac{\partial y}{\partial t}$$

$$\frac{\partial \sigma'}{\partial t} = \frac{\partial \sigma}{\partial t} - \gamma_w \frac{\partial h}{\partial t} + \gamma_w \frac{\partial y}{\partial t}$$

This then leads to the equation presented below:

$$\frac{\partial h}{\partial t} = \frac{1}{\gamma_w}\frac{\partial \sigma}{\partial t} + \frac{\partial y}{\partial t} + \frac{1}{m_v \gamma_w}\frac{\partial}{\partial y}\left(k \frac{\partial h}{\partial y}\right)$$

A drawback to this equation is that there are three dependent variables to solve for. This renders the problem highly non-linear and also reduces the number of numerical methods available to obtain a solution to this equation. It’s performance related to other finite strain formulations can be seen in the following sections.

2.3 Solution Methodology

Due to time restraints, a quick solution to the equation was necessary. Therefore a finite difference solution to the equation was developed. A model was built on Microsoft’s EXCEL 5.0 spreadsheet program which allowed a solution to the consolidation equation to be found. The results from this program were then compared to standard finite strain models which used finite element or finite difference as their solution methods. An iterative procedure was used to solve for the change in total stress and the change in height. This is because both variables are unknown at the start of a timestep. These variables were set to zero and then a change in head was then estimated. Once a change in head was calculated, a corresponding change in height and change in total stress could be calculated. These two variables were then substituted back into the start of the equation and the process was repeated until the two variables converged. This solution method allowed convergence of the equation and therefore the model could be tested against standard solutions.

It should be noted, however, that this equation does not lend itself to an easy solution methodology and so it is the authors recommendation that a better formulation be developed.

2.4 Prediction Scenario's

Prediction scenarios were taken from F.C. Townsend's "Large Strain Consolidation Predictions" paper. A total of four scenarios were presented in the paper but only three were used for comparison purposes. The reason for this was that one scenario required the model to simulate a boundary changing in elevation with time and it was not the capability of the model developed to perform this type of analysis. The three scenarios that were used for comparison purposes are presented below. Scenario's A, C, and D were used in the comparison process.

PREDICTION SCENARIOS

As shown in Fig. 1, predictions were requested for four disposal scenarios of increasing complexity. [Additional predictions not reported in this paper are contained in Townsend (1987).] The intent was to encompass a variety of boundary conditions, thereby examining the versatility of the various models as presented in the following:

1. Scenario A—Quiescent consolidation, uniform initial void ratio. Scenario A is a quiescent consolidation prediction of a single drained waste clay pond instantaneously placed at a uniform void ratio, e_o = 14.8 (initial solids content Si = 16%) having a height of 31.5 ft (9.6 m). This case is intended to simulate waste ponds into which thickened clays have been recently pumped and that subsequently consolidate due to self-weight stresses.

2. Scenario B—Stage filling, nonuniform initial void ratio. Scenario B is a prediction for a 23.6-ft (7.2-m) deep pond filled in two stages with two different initial void ratios. The pond will be filled in two six-month increments, separated by a six-month quiescent consolidation increment with the clay at an initial void ratio of 14.8 (Si = 16%) for the first filling increment and at a void ratio of 22.82 (Si = 11%) for the second filling increment. The filling rate is 0.0656 ft/day (0.02 m/d). This case is intended to simulate final waste clay ponds that

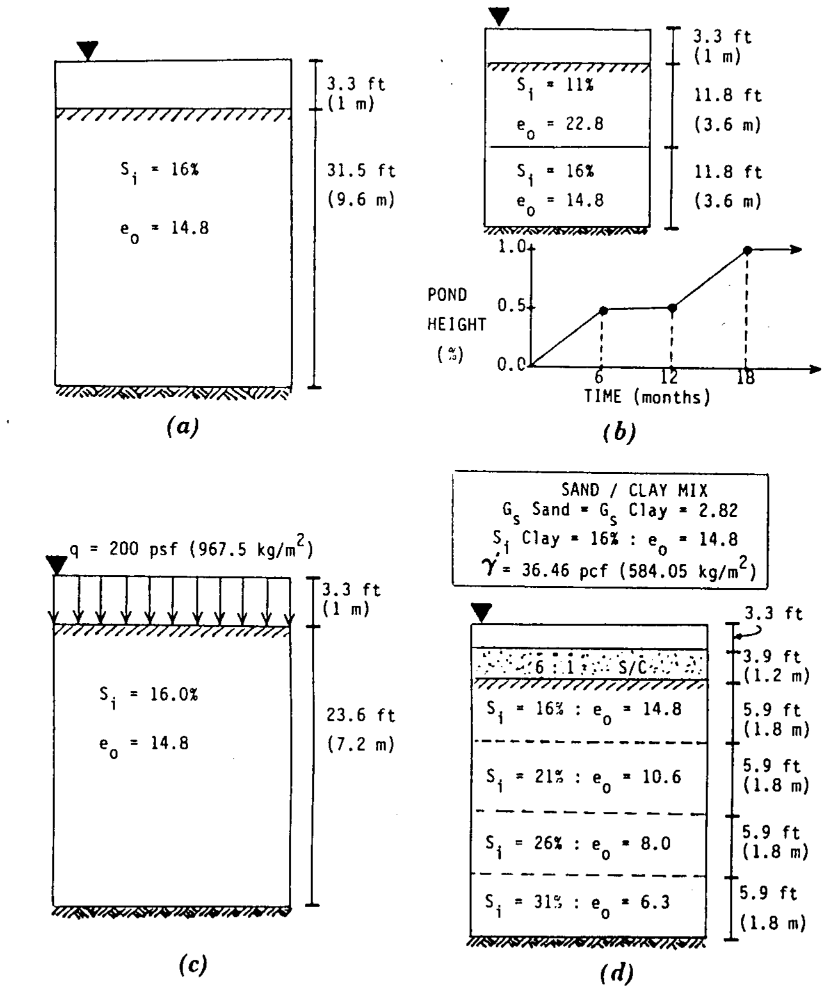


FIG. 1. Summary of Four Prediction Scenarios: (*a*) Scenario A: Quiescent Consolidation Uniform Initial Void Ratio; (*b*) Scenario B: Stage Filling Nonuniform Void Ratio; (*c*) Scenario C: Quiescent Consolidation Surcharge Loading Uniform Initial Void Ratio; (*d*) Scenario D: Two-Layer Quiescent Consolidation Sand/Clay Surcharge Nonuniform Initial Void Ratio

are filled intermittently with thickened clays pumped from an initial settling area.

3. Scenario C—Quiescent consolidation and surcharge loading of a pond having a uniform initial void ratio. Scenario C predicts the consolidation of a 23.6-ft (7.2-m) deep waste pond with a uniform initial void ratio of 14.8 (Si = 16%), subjected to surcharge of 200 psf (967.5 kg/m^2). This scenario simulates a young waste pond, so the void ratio profile is uniform. It is to be capped with a 200-psf (976.5-kg/m^2) surcharge, representing a four-foot (1.22-m) thick sand layer with a buoyant unit weight γ' = 50 pcf (800.9 kg/m^3).

4. Scenario D—Two-layer quiescent consolidation, sand/clay (S/C) sur-

charge and nonuniform initial void ratio. Scenario D is the prediction of a 23.6-ft (7.2-m) deep waste pond with a nonuniform initial void ratio profile varying from 14.8 to 6.28 (Si = 16–31%) that is subjected to a 142.2-psf (6.8-kPa) surcharge load by a 6:1 S/C cap, 3.9 ft (1.2 m) thick. The sand/clay cap has a different void ratio-effective stress-permeability relationship than the underlying waste clay and has a buoyant unit weight $\gamma' = 36.46$ pcf (584.05 kg/m^3). This scenario simulates a mature waste pond with a nonuniform initial void ratio profile that is reclaimed by placement of an S/C cap.

The capability of the clays to support surcharge loads assumed in scenarios C and D was not considered in the analyses.

Input Parameters and Useful Equations

The effective stress-void ratio-permeability relationships that were selected for the predictions are presented as the following equations:

Waste clay

$$e = A(\bar{\sigma})^{-B(\mathrm{psf})} \Rightarrow e = 15.07(\bar{\sigma})^{-0.22} \quad (1)$$

$$k = c(e)^{D}(\mathrm{ft/day}) \Rightarrow k = (0.8304\ E - 06)e^{4.65} \quad (2)$$

or

$$e = A(\bar{\sigma})^{-B}(\mathrm{kPa}) \Rightarrow e = 7.72(\bar{\sigma})^{-0.22} \quad (3)$$

$$k = c(e)^{D}(\mathrm{m/d}) \Rightarrow k = (0.2532\ E - 06)e^{4.65} \quad (4)$$

6:1 sand/clay mix (scenario D)

$$e = A(\bar{\sigma})^{-B}(\mathrm{psf}) \Rightarrow 32.5(\bar{\sigma})^{-0.24} \quad (5)$$

$$k = c(e)^{D}(\mathrm{ft/day}) \Rightarrow (0.4235\ E - 06)e^{4.15} \quad (6)$$

or

$$e = A(\bar{\sigma})^{-B}(\mathrm{kPa}) \Rightarrow 15.67(\bar{\sigma})^{-0.24} \quad (7)$$

$$k = C(e)^{D}(\mathrm{m/d}) \Rightarrow (0.1291\ E - 06)e^{4.15} \quad (8)$$

Conversion factors are as follows: 1 psf = 4.9 kg/m^2; and 1 m = 3.3 ft.

In addition, the following equations are useful for determining void ratios, solids contents, and unit weights:

$$e = \frac{G(1 - S)}{S} \quad (9)$$

$$S = \frac{G}{(e + G)} \quad (10)$$

$$\gamma' = \frac{S\gamma_w(G - 1)}{G + S(1 - G)}, \quad \text{clay only} \quad (11)$$

$$\gamma' = \frac{S\gamma_w(\mathrm{SCR} + 1)(G - 1)}{G(1 - S) + S(1 + \mathrm{SCR})}, \quad \text{sand/clay mix} \quad (12)$$

where e = void ratio; S = solids content clay; γ' = buoyant unit weight; G = specific gravity for both sand and clay ($G_{sand} = G_{clay}$); and SCR = sand/clay ratio. For example, a waste clay having a void ratio of 10.00 corresponds to a solids content of 22.00 (G = 2.82), a buoyant unit weight of 7.76 pcf (124.3 kg/m^3), an effective stress ($\bar{\sigma}$) of 6.45 psf (0.31 kPa), and a permeability of 0.04 ft/day (0.01 m/d).

TABLE 2. Summary of Predictions and Consolidation Model Origin

Predictor (1)	Affiliation (2)	Scenario Predicted A (3)	B (4)	C (5)	D (6)	Program description (7)
Carrier	Bromwell & Carrier, Inc.	X	X	X	0^a	Somogyi-based
		X			X	Approximate integration for magnitude
Trin et al.	Northwestern	X	X	X	0	Somogyi-based
Garlanger et al.	Ardaman & Assoc.	X				Piecewise linear (Olson)
			X	X		Somogyi-based
McVay et al.	University of Florida	X	X	X		Piecewise linear (Yong)
		X	X	X		Somogyi-based
		X		X	X	Closed-form integration for magnitude
Poindexter et al.	WES	X	0^a	X		Cargill-based
Long et al.	University of Conn.	X	X	X		Cargill-based
Yong et al.	McGill	X	X	X	0^a	Piecewise linear (Yong)
Feldkamp	—	X	X			Finite element
Pyke	TAGA	X	X	X	0^a	Piecewise linear

[a] Approximate solutions.

PREDICTIONS

As summarized in Table 2, nine different model users representing academia, government, and consulting firms provided predictions. The difficulty of the disposal scenarios posed is reflected in the number of prediction attempts. For scenario A, (quiescent conditions), ten predictions plus two magnitude-only estimates were provided. On the other hand, for scenario D (nonuniform, different material properties), four predictions (all using approximate methods) and two magnitude-only estimates were provided. Therefore, on this basis, it is concluded that a variety of programs are available for estimating consolidation magnitudes and rates under conditions of quiescent consolidation, stage filling, and surcharge loading of uniform and, in some cases, nonuniform waste ponds. However, for conditions involving materials with dissimilar compressibility relationships, program availability is limited.

2.5 Discussion of Results

The greatest difficulty in the modelling of the three scenario's was convergence problems and numerical instability encountered in the developed finite difference model. For the consolidation equation presented it was unknown how large a timestep should be used and what number of nodes were required to produce sufficiently accurate results. This had to be determined by a trial and error approach. The EXCEL model was modified so that the amount of change of key properties in the equation (pore-water pressure, void ratio, unit weight, and height) could be strictly monitored. The time increment allowed was also regulated so that if the model was having difficulty converging, the time step would automatically be reduced until a suitable time step could be found. In the finite difference solution to Terzaghi's consolidation equation, a variable named Beta is introduced that must be limited to be less than 1/2 for the calculated time increment to be accurate. Beta in the original equation is equal to $\frac{c_v \Delta t}{\Delta y^2}$ and in the new equation Beta was set to $\frac{\Delta t(1+e_o)}{\gamma_w a_v}$. It was guessed that this would be the controlling influence on the finite difference equation but during model testing this assumption was found to be untrue. The Beta variable could still be used to control time increments but could not guarantee an accurate solution. To estimate whether a timestep was accurate, tolerance levels were put in place for the maximum amount of change allowed to the key variables mentioned above. Timesteps were reduced to allow convergence within these tolerance limits.

A second problem encountered with convergence was the steep function input describing the relationship between void ratio and effective stress. Since a forward difference technique was used to estimate volume change, errors were great if the amount of change was not minimized. This is illustrated in Figure 4 showing the relationship between void ratio and effective stress that was used in the predictions.

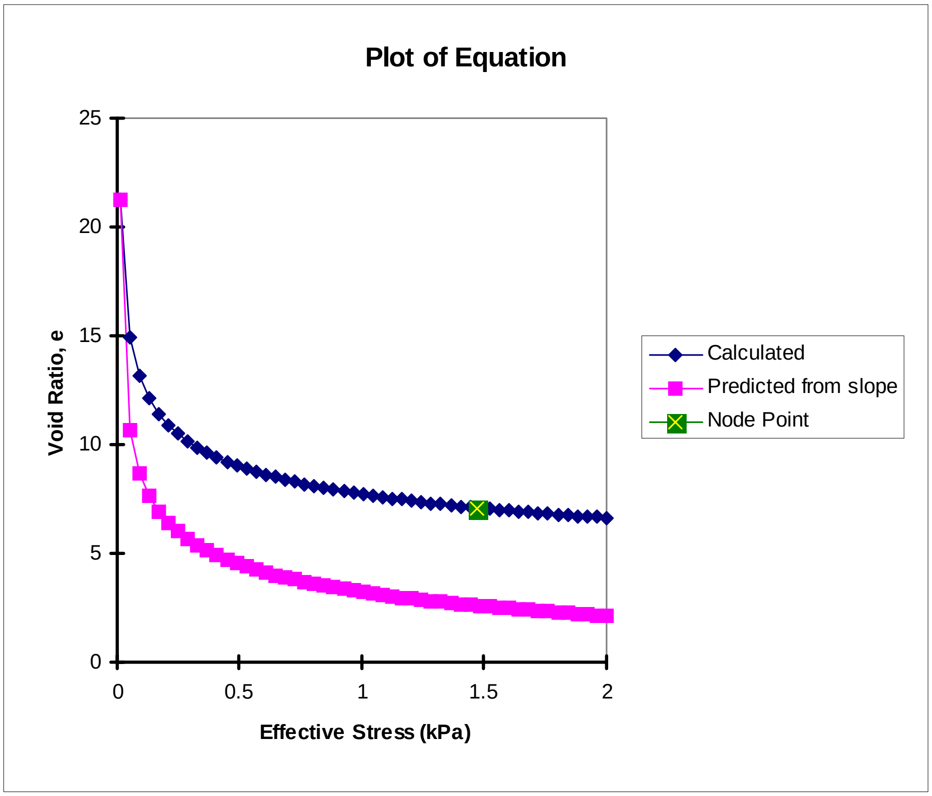


Figure 4 - Stress/Volume change relationship used in model

The relationship between permeability and void ratio created numerical instability in certain situations. This is illustrated in Figure 5 where a graph of the permeability for different void ratios can be seen. When a zero pressure boundary was introduced, this had the effect of immediately compressing the boundary element. Void ratio decreased and the permeability decreased by several orders of magnitude. This had the effect of trapping water inside the consolidation layer. It was found, however, that this problem could be overcome by increasing the number of nodes used in the analysis. The smaller element size then reduced the effect of this problem.

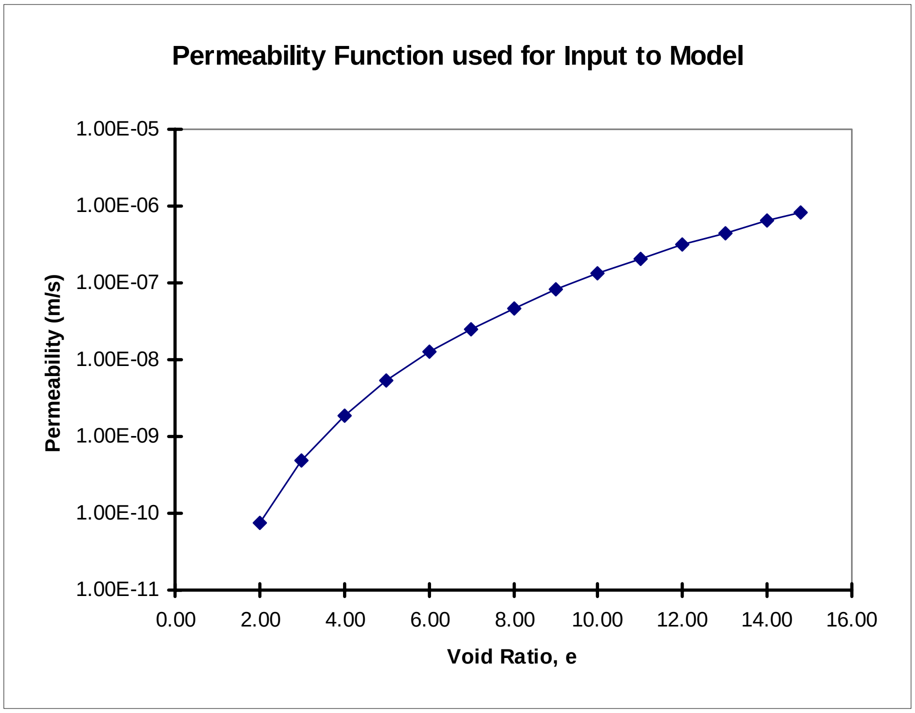


Figure 5 - Permeability function used for model

2.5.1 Scenario A

The finite difference model developed on EXCEL (the model is hereafter called "Squeezer") matched this scenario almost exactly with a minimum number of nodes. Only 20 nodes were used in the analysis and run times to a one-year solution were approximately 15 minutes. The results of Squeezer alongside data gathered from other prediction models by F.C. Townsend can be seen in the following figures.

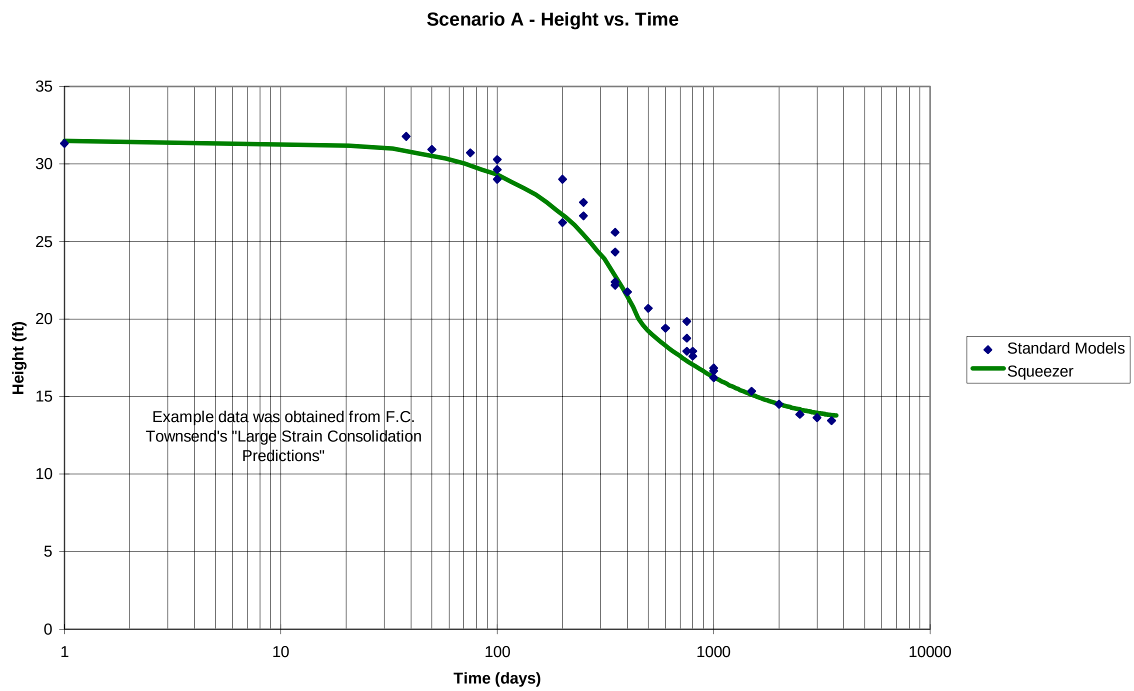


Figure 6 - Comparison of height of consolidation layer to other predictors

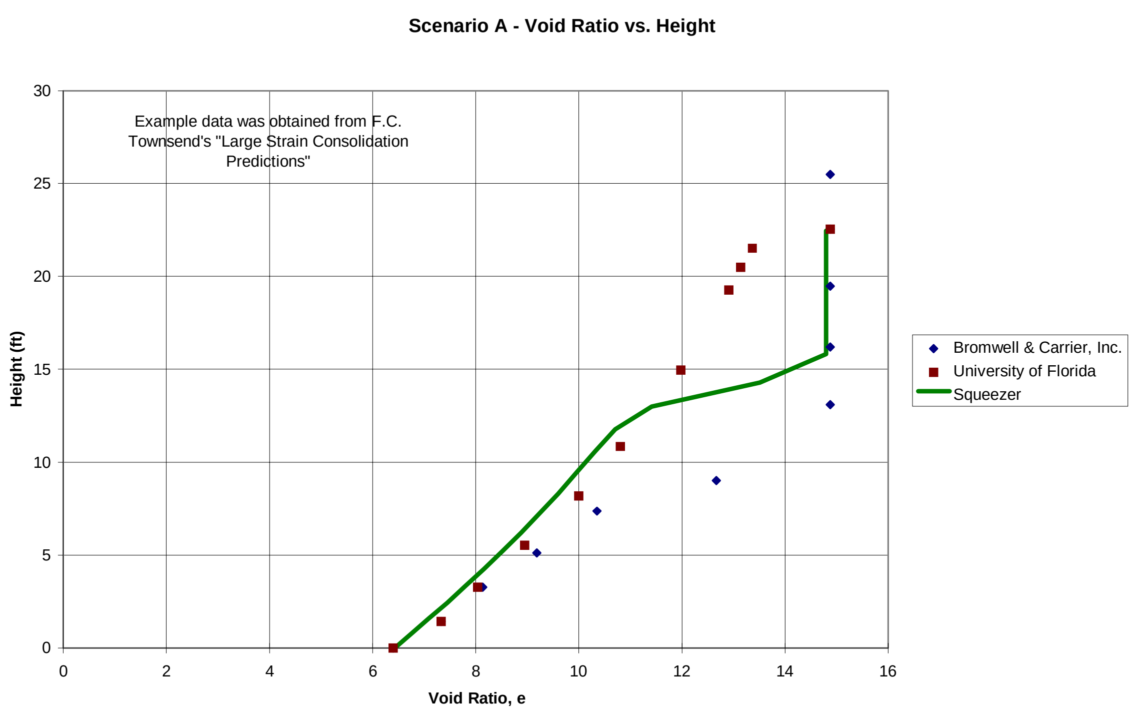


Figure 7 - One-year profile of void ratio among predictors

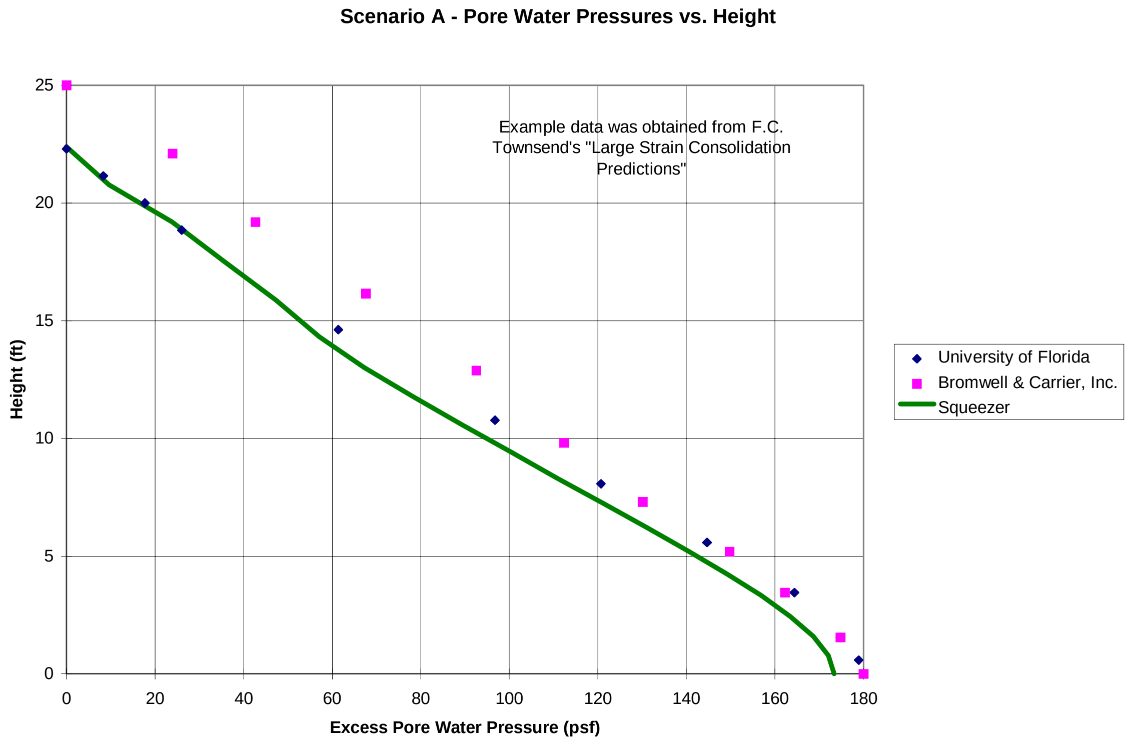


Figure 8 - One year profile of excess pore-water pressure among predictors

The slight disagreement between programs can be attributed to differing numerical techniques used by the predictors to obtain solutions. Overall the Squeezer model provided close correlation to industry standard programs.

2.5.2 Scenario C

The correlation between Squeezer and standard models was not as favorable for Scenario C. The difference seemed to be centered on the height prediction of the consolidation layer. The results from Squeezer and other predictors can be seen in Figure 9 .

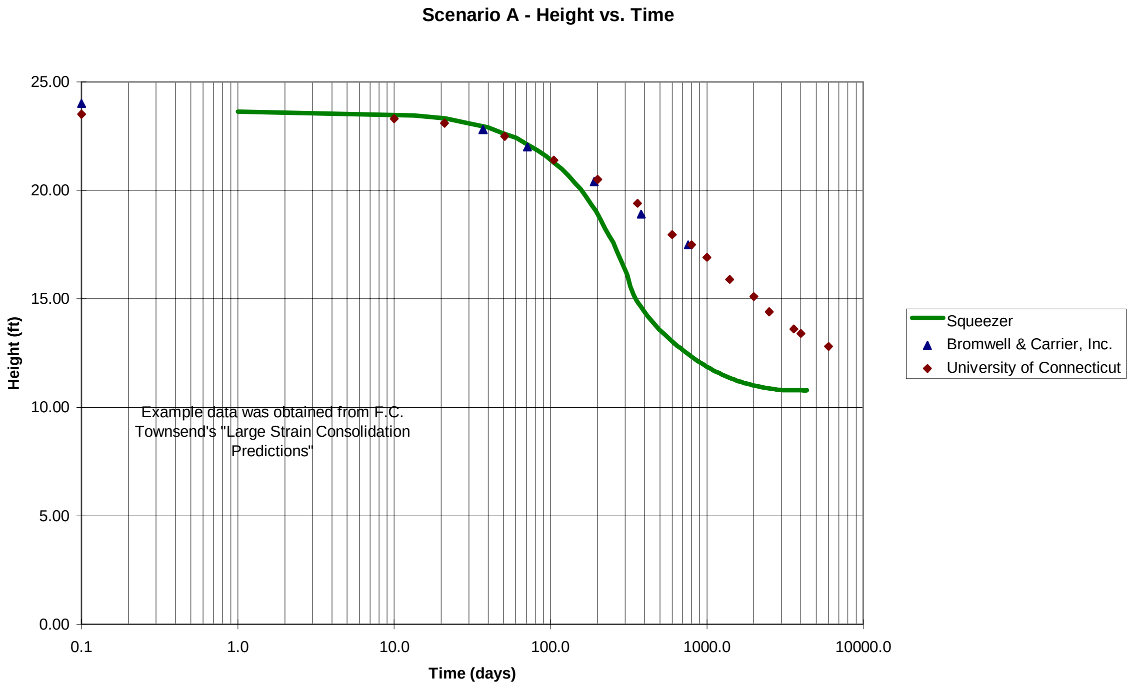


Figure 9 - Correlation of height of consolidation layer among predictors for Scenario C

Compression of the consolidation layer seemed to occur too quickly. From closer examination of the Squeezer model, this appears to be due to the forward difference prediction of volume change used by the model. the steepness of the void ratio vs. effective stress function caused the height to consistently be underpredicted. This is illustrated in Figure 9. The author lacked sufficient time to correct this problem but the problem may be minimized by using extremely small time steps.

The zero pressure upper boundary was a source of numerical instability for Scenario C. This caused a reduced permeability along the upper boundary and had the effect of trapping water in the soil as was previously described. To overcome this, the number of nodes used was increased from 20 to 200. This then yielded a reasonable solution at the expense of solution time. To achieve a one-year solution with 200 nodes consumed 3 days of computer time on a 486-66 computer. The long solution times as well as the uncertainty of the error cause the usefulness of the finite difference method to be closely examined. A faster solution method such as finite element would be recommended.

Notwithstanding the height predictions, the void ratio and pore-water pressure profiles were reasonable as shown in the following figures. Inaccuracy can be attributed to numerical problems of height prediction.

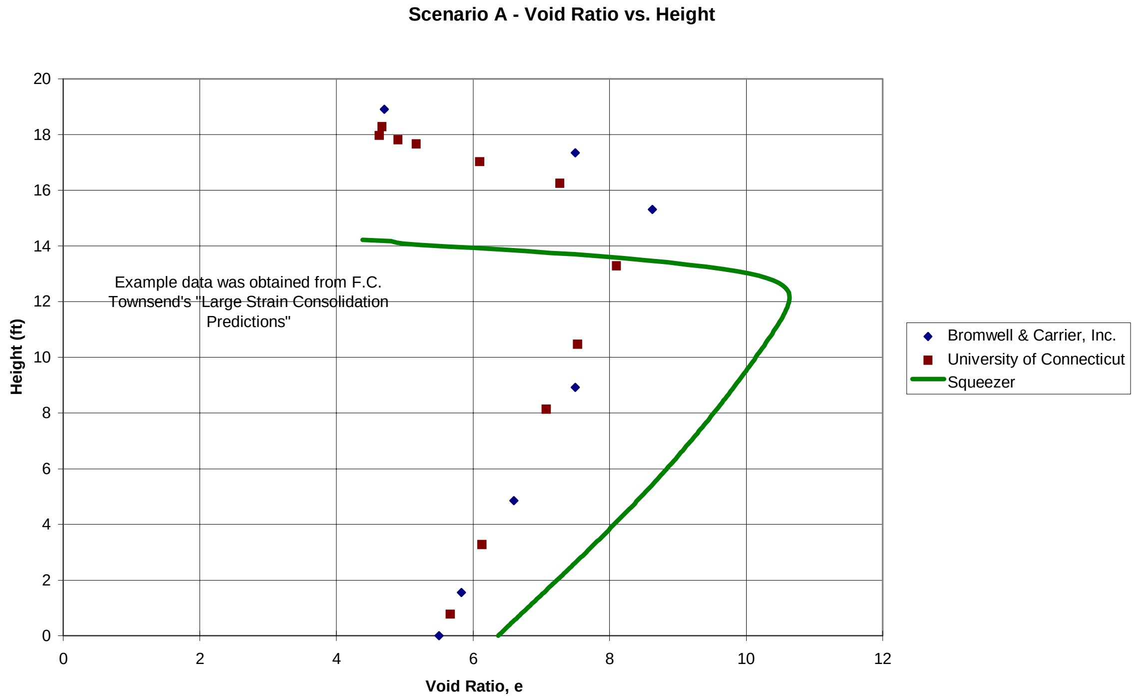


Figure 10 - One-year void ratio prediction for Scenario C

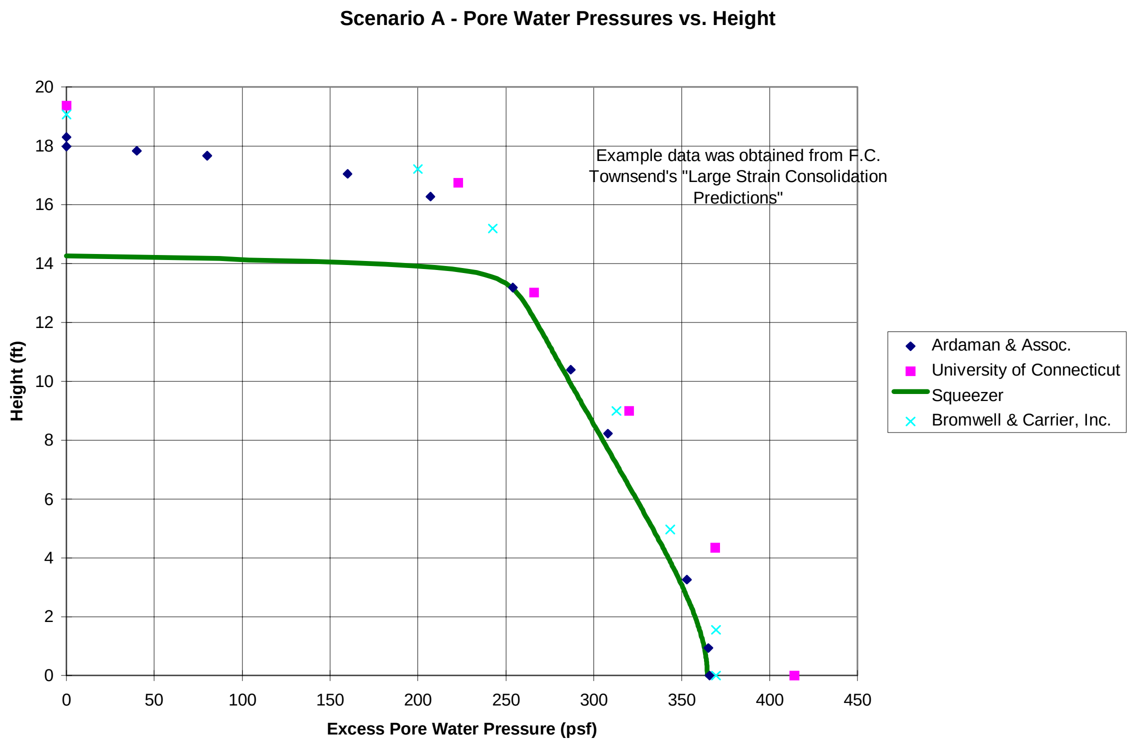


Figure 11 - One-year profile of pore-water pressure for Scenario C

2.5.3 Scenario D

The difference between standard predictors and the model Squeezer was significant for Scenario D. It is estimated that boundary effects between the differing soil layers are responsible for most inaccuracies. Increasing the number of nodes used in the analysis might possibly give better results. Since the solution presented used 200 nodes, it is suggested that 500 nodes may be needed for an accurate solution. Sufficient time to confirm this estimate was not available. The comparison of results to standard predictors are shown in the following figures.

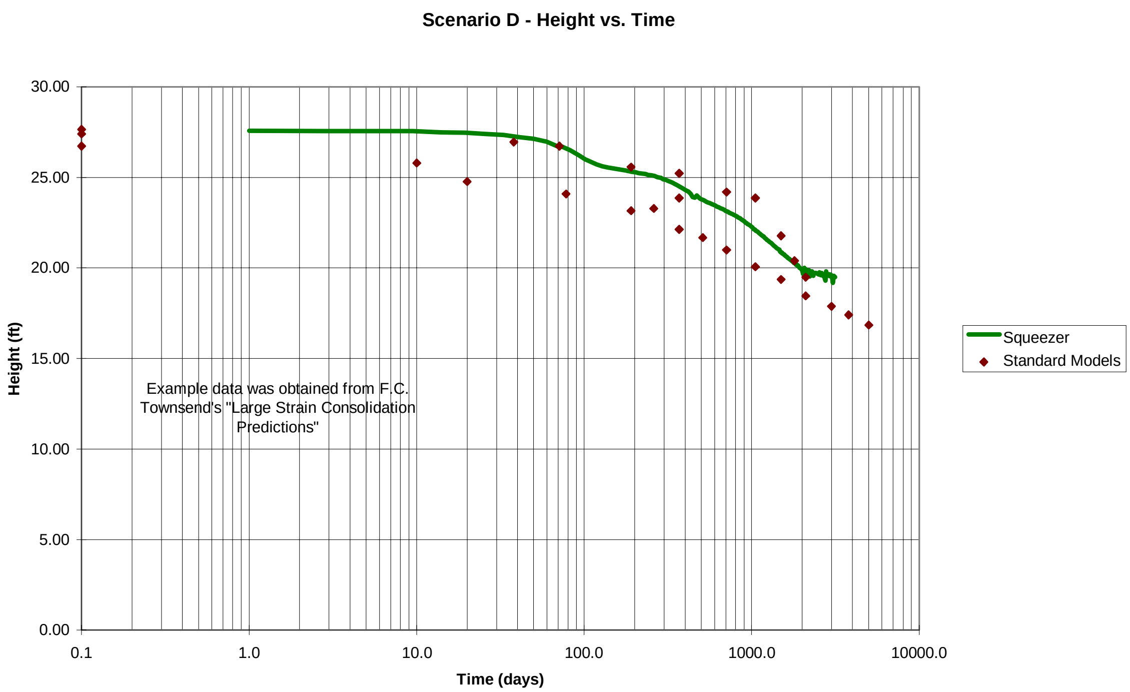


Figure 12 - Comparison of height estimations among predictors

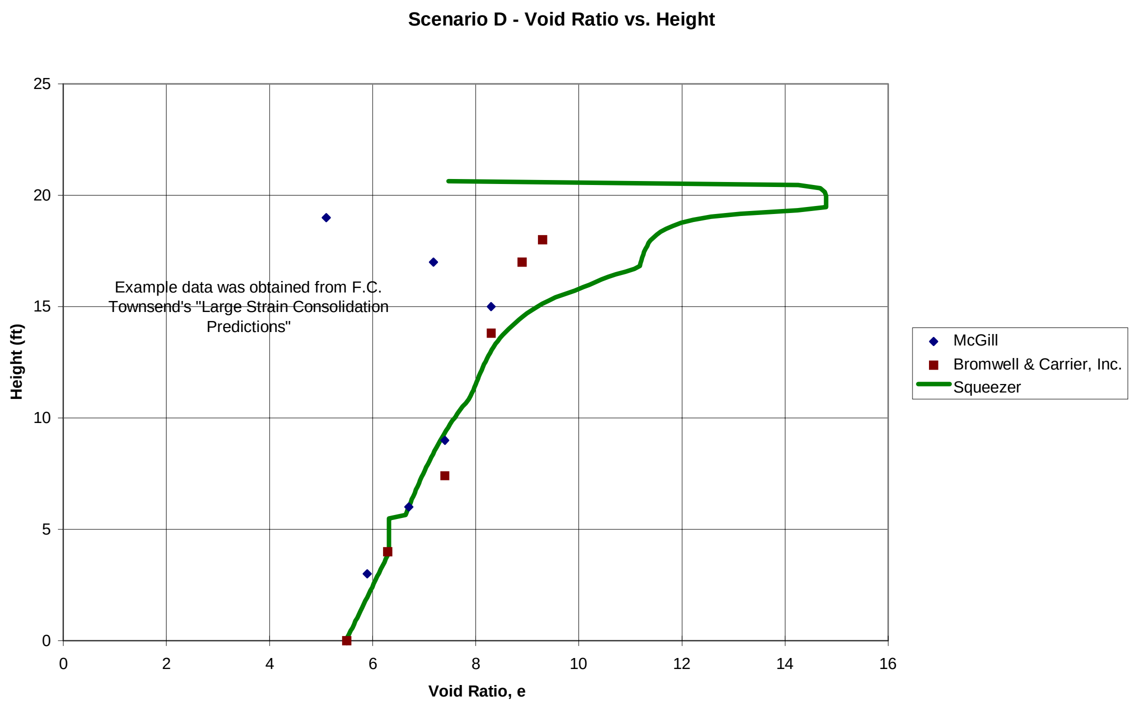


Figure 13 - One-year profile of void ratio for Scenario D

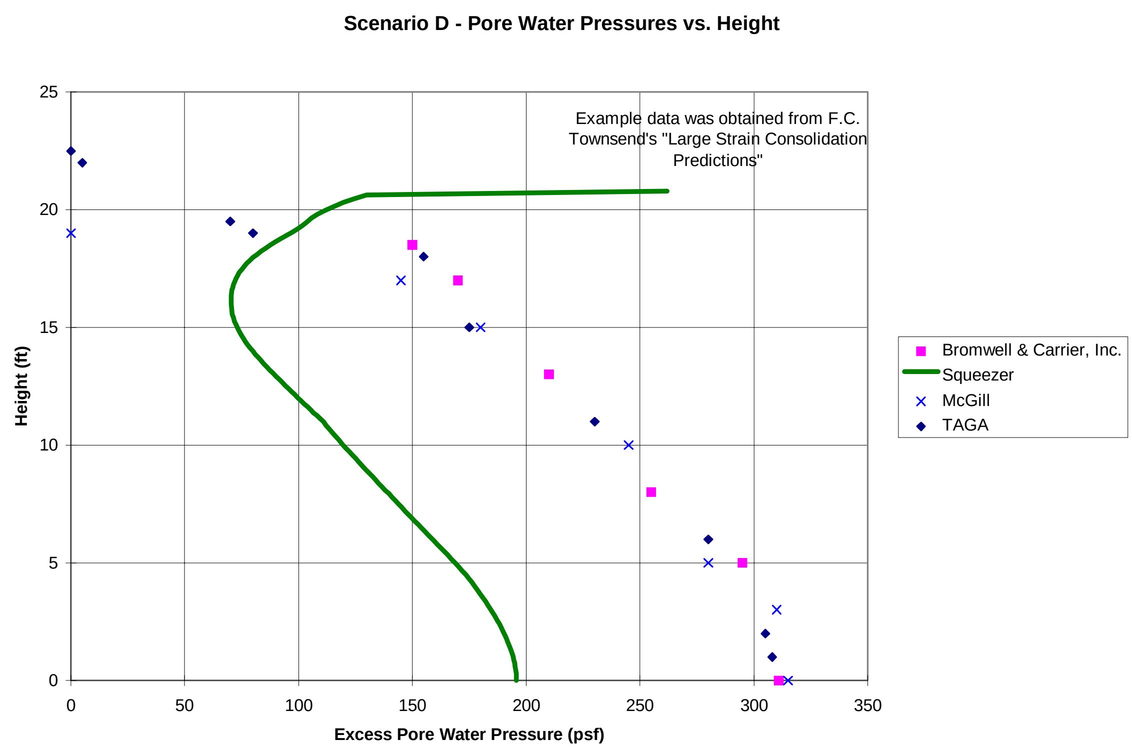


Figure 14 - One-year profile of excess pore-water pressure for Scenario D

2.6 Conclusion

While allowing for reasonable solution to simple problems, it is the estimation of the author that a better formulation to the consolidation problem be found. The existing formulation created significant convergence problems and extremely long run-times to provide reasonable solutions. Formulations that may be adapted to the finite element method of numerical solutions are recommended. While adequate for the solution of simple partial differential equations, the solution of complex partial differential equations by the finite difference method becomes time consuming.

Solutions of surcharge boundary loading conditions were quite sensitive to the number of layers used in the analysis yet an accurate way of determining the number of layers needed was not found. It is the authors recommendation that further research be done in this field.

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